MATH 207(c)–Analysis IV

G. Moschidis

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1. Calculate the Fourier transform of the following functions (the residue theorem might be useful for a few cases, but not in all of them):

(a) $f(x) = \dfrac{1}{x^2 - 2x + 2}$
Hint: If you want, you can avoid lengthy computations by using the properties of the Fourier transform and the fact that, as we computed in class last week,

$$\mathcal{F}[\frac{1}{x^2+1}](a) = \sqrt{\frac{\pi}{2}}e^{-|a|}.$$

(b) $f(x) = \dfrac{x}{x^4+1}$.

(c) $f(x) = e^{-|x|}$ (in this case, you should get $\hat{f}(a) = \sqrt{\frac{2}{\pi}}\frac{1}{a^2+1}$; there are a few different ways to obtain this result).

2. Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous function satisfying

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty \quad \text{and} \quad \int_{-\infty}^{+\infty} |x \cdot f(x)|\, dx < +\infty.$$

Show that the Fourier transform of $g(x) = x \cdot f(x)$ is well-defined and satisfies

$$\hat{g}(a) = i\frac{d}{da}\hat{f}(a).$$

Use this result to calculate the Fourier transform of $f(x) = xe^{-|x|}$.

3. Consider the following ordinary differential equation:

$$y''(x) + 2y'(x) + 5y(x) = e^{-|x|}, \quad x \in \mathbb{R}. \tag{1}$$

(a) By considering the Fourier transform of the above expression, find an expression for $\mathcal{F}[y](a)$.

(b) By considering the inverse Fourier transform of the expression for $\mathcal{F}[y](a)$, determine a solution $y(x)$ to (1).

Remark. The equation (1) is 2^{nd} order and has no initial or boundary value conditions, so one would expect to have a 2-dimensional space of solutions, not just a single solution. This is indeed true; however, among the solutions in this 2 dimensional space, only one goes to 0 as $x \to \pm\infty$; this is the only one for which the Fourier transfor $\mathcal{F}[y]$ is well-defined (since for the others the corresponding integral does not converge), and hence, this is exactly the one which is "selected" by our method above. In other words, applying the Fourier transform to (1) implicitly requires assuming that $y(x) \to 0$ as $x \to \pm\infty$ in order for the transform to be well defined, and this corresponds to imposing two boundary conditions at $x = \pm\infty$.

4. Use the various properties of the Fourier transform (i.e. about translations, re-scalings, frequency shifts etc, as well as the property proved in Ex. 2) together with the table of Fourier transforms given at the end of the sheet to calculate the Fourier transforms of the following functions:

(a) $f(x) = \dfrac{e^{ix}}{\beta^2 + \sigma^2 x^2}$, $\beta, \sigma \in \mathbb{R} \setminus \{0\}$.

(b) $g(x) = x^2 e^{-\beta^2 x^2}$, $\beta \in \mathbb{R} \setminus \{0\}$.

(c) $h(x) = \dfrac{x^2 - 2x + 1}{(x^2 - 2x + 2)^2}$.

5. Using the properties of the Laplace transform that we saw in class, show that the indicated $\gamma_0 \in \mathbb{R}$ is an abscissa of convergence and compute the Laplace transforms of the following functions $f : [0, +\infty) \to \mathbb{C}$:

(a) $f(t) = (t+1)^3$, $\gamma_0 = 0$.

(b) $f(t) = \sin(\omega t)$ (where $\omega \in \mathbb{R}$), $\gamma_0 = 0$.

(c) $f(t) = t^2 \cos(\omega t)$ (where $\omega \in \mathbb{R}$), $\gamma_0 = 0$.

(d) $f(t) = \cosh(\omega t)$ (where $\omega \in \mathbb{R}$), $\gamma_0 = |\omega|$.

6. For two piecewise continuous functions $f, g : [0, +\infty) \to \mathbb{C}$, we define their convolution $f * g : [0, +\infty) \to \mathbb{C}$ by the relation

$$f * g(t) \doteq \int_0^t f(s) g(t-s)\, ds.$$

(a) Show that the above definition coincides with the usual definition of the convolution of $f, g : \mathbb{R} \to \mathbb{C}$ if we assume that f, g are extended on $(-\infty, 0)$ by the requirement that they are identically 0 there.

(b) Show that the Laplace transform of $f * g$ satisfies

$$\mathcal{L}[f * g](z) = \mathcal{L}[f](z) \cdot \mathcal{L}[g](z)$$

for any $z \in \mathbb{C}$ for which $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined *(Hint: Write down the expression for the Laplace transform and use the (trivial) identity $e^{-zt} = e^{-zs} e^{-z(t-s)}$).*

15.5 Table de transformées de Fourier

	$f(y)$	$\mathfrak{F}(f)(\alpha) = \hat{f}(\alpha)$
1	$f(y) = \begin{cases} 1 & \text{si } \lvert y\rvert < \lvert b\rvert \\ 0 & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\sin(\lvert b\rvert\, \alpha)}{\alpha}$
2	$f(y) = \begin{cases} 1 & \text{si } b < y < c \\ 0 & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{e^{-i\, b\, \alpha} - e^{-i\, c\, \alpha}}{i\, \alpha\, \sqrt{2\pi}}$
3	$f(y) = \begin{cases} e^{-\omega\, y} & \text{si } y > 0 \\ 0 & \text{sinon} \end{cases} \quad (\omega > 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{1}{(\omega + i\, \alpha)}$
4	$f(y) = \begin{cases} e^{-\omega\, y} & \text{si } b < y < c \\ 0 & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \frac{e^{-(\omega + i\, \alpha) b} - e^{-(\omega + i\, \alpha) c}}{(\omega + i\, \alpha)}$
5	$f(y) = \begin{cases} e^{-i\, \omega\, y} & \text{si } b < y < c \\ 0 & \text{sinon} \end{cases}$	$\hat{f}(\alpha) = \frac{1}{i\sqrt{2\pi}} \frac{e^{-i(\omega+\alpha) b} - e^{-i(\omega+\alpha) c}}{\omega + \alpha}$
6	$f(y) = \frac{1}{y^2 + \omega^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{\pi}{2}} \frac{e^{-\lvert\omega\, \alpha\rvert}}{\lvert\omega\rvert}$
7	$f(y) = \frac{e^{-\lvert\omega\, y\rvert}}{\lvert\omega\rvert} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2 + \alpha^2}$
8	$f(y) = e^{-\omega^2 y^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \frac{1}{\sqrt{2}\, \lvert\omega\rvert} e^{-\frac{\alpha^2}{4\omega^2}}$
9	$f(y) = y\, e^{-\omega^2 y^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \frac{-i\, \alpha}{2\sqrt{2}\, \lvert\omega\rvert^3} e^{-\frac{\alpha^2}{4\omega^2}}$
10	$f(y) = \frac{4\, y^2}{(\omega^2 + y^2)^2} \quad (\omega \neq 0)$	$\hat{f}(\alpha) = \sqrt{2\pi} \left(\frac{1}{\lvert\omega\rvert} - \lvert\alpha\rvert\right) e^{-\lvert\omega\, \alpha\rvert}$